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Calculation of the Propagator for a Time-Dependent Damped, Forced Harmonic Oscillator Using the Schwinger Action Principle

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A calculation of the quantum mechanical propagator for a general time-dependent one-dimensional damped, forced harmonic oscillator based on a direct application of the Schwinger action principle is presented. Contact with an earlier general method of calculation is made and in particular two previously found results are recovered from our general expression. The apparent dependence of one of them upon some unphysical parameters is clarified. The method presented here can be applied to any system described by a quadratic Hamiltonian and is an immediate extension of the calculation for the propagator of a simple harmonic oscillator with constant frequency.

1. INTRODUCTION

The problem of the calculation of the quantum mechanical propagator $K(q'', t''; q', t')$ (transformation function, Green's function) for a system described by a general quadratic Lagrangian is solved in principle through the use of the Van Vleck–Pauli formula which reduces to ($\hbar = 1$)

$$K(q'', t''; q', t') = \left(\frac{i}{2\pi} \frac{\partial^2 S}{\partial q' \, \partial q''} \right)^{1/2} \exp\left[iS(q''t'', q't') \right] \tag{1}$$

in the one-dimensional case. The problem is thus shifted to the calculation of the classical action S as a function of the end points q', t' and q'', t'' with $S = \int_{t'}^{t''} L(t)\, dt$. Such calculation is not completely straightforward especially when the parameters of the Lagrangian depend explicitly on time.

Recently there has been a renewal of the interest in the calculation of the quantum mechanical propagator for a one-dimensional harmonic oscillator with time-dependent frequency and damping coefficient acted upon by an external time-dependent force (Khandekar and Lawande, 1978; Dodonov et al., 1979). The calculation of Khandekar and Lawande (1978), valid for a constant damping coefficient, is based on first finding an explicitly time-dependent constant of motion which is subsequently used to define a canonical transformation. The generating function of such canonical transformation is then found and shown to be simply related to the action S which is finally substituted in equation (1). The final expression for the propagator K has an explicit dependence upon the end point coordinates q' and q'', while the time dependence is given implicitly through the evaluation of two auxiliarly functions at the initial and final times. This is a common feature of all those calculations where the explicit time dependence of the frequency and damping coefficient are not specified. One of the time-dependent auxiliary functions used by Khandekar and Lawande satisfies a nonlinear second-order differential equation (two constants of integration), while the other is obtained by integrating once with respect to time certain function of the first auxiliary function (one additional constant of integration). The boundary conditions of these auxiliary functions are not specified by the construction of the propagator. This situation leads to the uncomfortable feeling that the propagator depends on these three arbitrary constants of integration. At least, the explicit form for K given by Khandekar and Lawande allows us to verify that the final result is indeed independent of the arbitrary integration constant associated with the second auxiliary function referred to above. However, it is far from clear which is the situation regarding the dependence of the propogator K upon the remaining two arbitrary constants of integration.

The calculation of the propagator carried by Dodonov et al. (1979) is valid for any damping coefficient and is based on a previously established connection between the integrals of the motion of a quantum system and its propagator (Dodonov et al., 1975). In fact, Dodonov et al. (1975) show that the action of the initial position and initial momentum operators upon the transformation function can be expressed in terms of simple operations on the initial coordinates of the propagator. The initial position and momentum operators, regarded here as integrals of the motion, are subsequently expressed as functions of the dynamical operators at the final time. In this way a set of partial differential equations for the Green's function in terms of the initial and final coordinates is obtained. Such procedure can be explicitly carried out for a general time-dependent quadratic Larangian in virtue of the linearity of the corresponding equations of motion. The resulting partial differential equations can be readily integrated and together with the Schrödinger equation completely determine the propagator. Again, the final expression for K is explicit in the initial and final coordinates and the time dependence is incorporated via two auxiliary functions which now satisfy a linear second-order differential equation. This time the boundary conditions for such auxiliary functions are uniquely defined by choosing the integrals of motion to be the initial position and momentum operators.

It is well known that the Green's function for a general quadratic Lagrangian with an external time-dependent force can also be obtained by using Feynman path integral formulation of quantum mechanics (Feynman and Hibbs, 1965). Calculations along these lines for some particular cases have been carried by Papadopoulos (1974).

In this work we present a calculation of the propagator for a one-dimensional damped-force harmonic oscillator with arbitrary time-dependent parameters which is based in an application of the Schwinger action principle for quantum mechanics. We recover the result of Dodonov et al. (1979) and show that their method is both in spirit and in application closely related to the calculation of the Green's function according to Schwinger's ideas.

Section 2 contains a brief review of the action principle in quantum mechanics (Schwinger, 1951, 1970) together with the calculation of the propagator for the harmonic oscillator with constant frequency which will serve as a model for the time-dependent case. A discussion of the integrability conditions for the differential equations which determine the propagator is given in the general case. In this section we also show how the relations between the integrals of motion and the Green's function discovered by Dodonov et al. (1975) can be obtained as a direct application of the action principle.

In Section 3 we extend the above-mentioned propagator calculation to the damped time-dependent harmonic oscillator case. The external time-dependent force is subsequently introduced via another application of the action principle and finally we recover the result of Dodonov et al. (1979). In this section we also make contact with the calculation of Khandekar and Lawande (1978). Our way of obtaining their expression for the propagator clearly shows that this result is indeed independent of any arbitrary integration constant. Some details regarding the calculations in this section are given in the Appendix.

Finally, Section 4 contains a brief summary and discussion of our work.

2. THE ACTION PRINCIPLE IN QUANTUM MECHANICS

The basic quantity in the quantum mechanical description of a system can be considered the transformation function $K(a'', t''; b', t') =$

$\langle a'', t''|b', t'\rangle$ which connects the representation in terms of a complete set of commuting operators having eigenvalues b' at time t' with the corresponding representation in terms of another complete set of commuting operators having eigenvalues a'' at the later time t''. This transformation function can also be regarded as the matrix elements, in a mixed basis, of the evolution operator $U(t'', t')$ of the system. In fact, $\langle a'', t''|b', t'\rangle = \langle a''|U(t'', t')|b'\rangle$, so that the transformation function contains all the dynamical information of the system.

In the literature we find two formulations of quantum mechanics where the basic object is taken to be the above-mentioned transformation function. One of them is Feynman path integral method (Feynman, 1948) which gives an expression of the transformation function in terms of the exponential of the classical action summed over all possible trajectories between the initial and final points in configuration space.

An alternative formulation is the Schwinger action principle, which we now briefly review. This principle is a differential characterization of the transformation function. It states that any conceivable infinitesimal variation in $\langle a'', t''|b', t'\rangle$ is given by the corresponding matrix elements of the variation of a single quantum mechanical operator

$$\delta\langle a'', t''|b', t'\rangle = i\langle a'', t''|\delta\left(\int_{t'}^{t''} L\,dt\right)|b', t'\rangle \tag{2}$$

which is the action operator. This operator is defined as the time integral of the Lagrangian operator L of the system. The Lagrangian is written in first order form and depends on the canonical variables q_i, p_i of the system in the following way

$$L = \frac{1}{2}\left(p_i\frac{dq_i}{dt} + \frac{dq_i}{dt}p_i\right) - H(q_i, p_i) \tag{3}$$

Here $q_i(t)$ and $p_i(t)$ are Hermitian position and momentum operators, respectively, and the Hamiltonian operator H contains all the dynamical information relevant to the system.

The next point to be specified is the operator character of the variations $\delta q_i(t)$ and $\delta p_i(t)$ that we are going to encounter when we vary the Lagrangian. For the application we have in mind here it is enough to consider only what Schwinger calls variables of the first kind. The variations appropriate to such variables are just pure numbers and therefore commute with all other operators.

From now on we will restrict ourselves to the one-dimensional situation and we will work in a complete and orthonormal coordinate basis defined by the eigenvectores $|q', t'\rangle$ of the position operator $q(t')$:

$$q(t')|q', t'\rangle = q'|q', t'\rangle \tag{4}$$

Leaving aside for the moment possible changes in the dynamics of the system, which would be reflected in changes on the parameters of the Hamiltonian, we realize that the only changes that we can produce on the transformation function are those related to alterations in the initial and final descriptions of the system: q', t', q'', t''. The related variable changes $\delta q', \delta t', \delta q'', \delta t''$ will induce a change in the states of the system through an infinitesimal unitary transformation in such a way that

$$\delta|q', t'\rangle = -iG(t')|q', t'\rangle \tag{5}$$

The infinitesimal Hermitian operator $G(t')$ is the generator of the corresponding unitary transformation at the time t' and depends on the changes $\delta q', \delta t'$ together with functions of the dynamical variables $q(t'), p(t')$. In this way we can write

$$\delta\langle q'', t''|q', t'\rangle = i\langle q'', t''|\big(G(t'') - G(t')\big)|q', t'\rangle \tag{6}$$

which allows us to express the action principle in operator form

$$\delta\int_{t'}^{t''} L\,dt = G(t'') - G(t') \tag{7}$$

after comparing equations (2) and (6). Equation (7) tells us that, for a given dynamics, the changes of the action operator depend only on the end points. The explicit calculation of the variation of the left-hand side of (7) will allow the identification of the generators G and the obtainment of the equations of motion for the dynamical variables. To this end it is convenient to think of the variables q, p, and t as being parametrized by an auxiliary variable τ which is kept fixed at the end points. In this way

$$\delta\int_{t'}^{t''} L\,dt = \delta\int_{\tau'}^{\tau''} \frac{1}{2}(p\,dq + dq\,p) - H\,dt \tag{8}$$

The fact that the variations δq and δp are pure numbers makes this calculation very similar to the analogous one in classical mechanics. The result is

$$\delta\int_{t'}^{t''} L\,dt = (p\,\delta q - H\,\delta t)\Big|_{t'}^{t''} + \int_{t'}^{t''} dt\left(\dot{q}\,\delta p - \dot{p}\,\delta q - \delta H + \frac{dH}{dt}\,\delta t\right) \tag{9}$$

which must depend only on the end points according to (7). The second term of the right-hand side of equation (9) clearly depends on the detailed history between the end points and then must be set equal to zero for arbitrary variations $\delta q(t)$, $\delta p(t)$, and $\delta t(t)$. This is achieved by setting

$$\begin{aligned}\delta H &= \dot{q}\,\delta p - \dot{p}\,\delta q + \frac{dH}{dt}\,\delta t \\ &= \frac{\partial H}{\partial p}\,\delta p + \frac{\partial H}{\partial q}\,\delta q + \frac{\partial H}{\partial t}\,\delta t \end{aligned} \tag{10}$$

where we have explicitly written the change of H in the last line of (10). The identification of the coefficients of the independent variations in (10) leads to the equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} \tag{11a}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{11b}$$

$$\dot{H} = \frac{\partial H}{\partial t} \tag{11c}$$

Furthermore, the first term of the right-hand side of (9) provides the identification of the generator of the end point transformations as

$$G(t) = p(t)\,\delta q(t) - H(t)\,\delta t \tag{12}$$

The general rule for the change of any operator X under an infinitesimal unitary transformation generated by G,

$$\delta X = \frac{1}{i}[X, G] \tag{13}$$

together with the possibility of performing independent coordinate and time displacements leads to the corresponding statements

$$i\frac{\partial F}{\partial q} = [F, p] \tag{14}$$

$$\frac{dF}{dt} - \frac{\partial F}{\partial t} = \frac{1}{i}[F, H] \tag{15}$$

which are valid for any function $F(q, p, t)$.

From equation (14) applied to the function $F = q$ we obtain the basic commutation relation

$$[q, p] = i \tag{16}$$

Equation (15) should be consistent with the equations of motion (11). It is immediate to verify that $F = p$ and $F = H$ substituted in the relation (15) reproduce indeed the equations of motion (11b) and (11c). To recover (11a) we need to identify $-q$ as the generator of changes in the momentum, which means that

$$\frac{\partial F}{\partial p} = -\frac{1}{i}[F, q] \tag{17}$$

This relation can be viewed as consequence of the basic commutator (16) but can also be obtained following steps completely analogous to those which led to expression (14) starting from the transformation function in the momentum basis $|p't'\rangle$ and using an appropriate version of the Lagrangian (3).

For the purposes of application to the calculation of the transformation function for systems with quadratical Hamiltonians it will be enough to consider equation (6) together with the explicit form of the generator (12) and the equations of motion (11). The basic idea is that equation (6) can be directly integrated for quadratical systems. This is because the linearity of the resulting equations of motion permits us to express $q(t)$ and $p(t)$, in terms of the operators $q(t')$ and $q(t'')$ whose action upon the corresponding eigenvectors $|q', t'\rangle$ and $|q'', t''\rangle$ is readily known. By doing this, equation (6) is reduced to a numerical relation which can be subsequently integrated. In order to illustrate this procedure in the simplest possible terms we review the calculation of the propagator for a harmonic oscillator defined by

$$H = \tfrac{1}{2}p^2 + \frac{\omega^2}{2}q^2 \tag{18}$$

where the frequency ω is constant. The equations of motion for the operators are

$$\dot{q} = p \tag{19a}$$

$$\dot{p} = -\omega^2 q \tag{19b}$$

which imply the following linear second-order differential equation for q:

$$\ddot{q} + \omega^2 q = 0 \tag{20}$$

The general solution of equation (20) is a linear combination of the functions $\sin\omega t$ and $\cos\omega t$ which contains two independent constants of integration which must be taken as operators in our case. Choosing such constants to be the operators $q(t')$ and $q(t'')$ we finally obtain

$$q(t)=\frac{1}{\sin\omega T}\big(q(t'')\sin\omega(t-t')+q(t')\sin\omega(t''-t)\big) \tag{21}$$

where $T=t''-t'$. From (21) and (19a) we find

$$p(t)=\frac{\omega}{\sin\omega T}\big(q(t'')\cos\omega(t-t')-q(t')\cos\omega(t''-t)\big) \tag{22}$$

Now we are ready to calculate the matrix element appearing in the right-hand side of equation (6)

$$\begin{aligned}&\delta\langle q'',t''|q',t'\rangle\\&\quad=i\langle q'',t''|\Big(p(t'')\,\delta q''-p(t')\,\delta q'-\frac{\delta T}{2}\big(p^2(t'')+\omega^2q^2(t'')\big)\Big)|q',t'\rangle,\end{aligned} \tag{23}$$

which we have rewritten for our particular case taking advantage of the fact that $H(t'')=H(t')$ in such a way that the relevant time variable is T. All matrix elements can be readily calculated in terms of the basic quantity $\langle q'',t''|q',t'\rangle$ except for the term proportional to $q(t')q(t'')$ which appears in the contribution $(1/2)\,\delta Tp^2(t'')$. In order to rewrite this term in such a way that both operators act on the appropriate eigenvectors we need the corresponding commutator which is

$$[q(t''),q(t')]=-\frac{i}{\omega}\sin\omega T \tag{24}$$

This result is obtained from the basic relation $[q(t''),p(t'')]=i$ when $p(t'')$ is expressed in terms of equation (22). After using (24) equation (23) reduces to

$$\begin{aligned}\delta\ln\langle q'',t''|q',t'\rangle=i\delta\bigg[&\frac{i}{2}\ln\sin\omega T\\&+\frac{\omega}{2}(q''^2+q'^2)\cot\omega T-\frac{q''q'\omega}{\sin\omega T}\bigg]\end{aligned} \tag{25}$$

where the changes on the right-hand side refer to $\delta q''$, $\delta q'$, and δT.

Equation (25) can be integrated immediately and the constant of integration that arises is determined by the normalization condition $\lim_{t''\to t'}\langle q'', t''|q', t'\rangle = \delta(q''-q')$. The final result is the well-known expression

$$\langle q'', t''|q', t'\rangle = \left(\frac{\omega}{2\pi i \sin\omega T}\right)^{1/2} \times \exp\left[\frac{i\omega}{2}\cot\omega T\left(q''^2 + q'^2 - \frac{2q'q''}{\cos\omega T}\right)\right] \tag{26}$$

In the above example we see that the equation (23) for the variation of the propagator can be integrated as shown in (25) after all the matrix elements are calculated. This means that some integrability conditions are satisfied and we are going to show how this works in the general situation. From (6) and the explicit form of the generators we know that

$$\delta\langle q'', t''|q', t'\rangle = i\langle q'', t''|p(t'')\,\delta q'' - p(t')\,\delta q' - H(t'')\,\delta t'' + H(t')\,\delta t'|q', t'\rangle \tag{27}$$

which imply the following partial differential statements:

$$\frac{\partial}{\partial q''}\langle q'', t''|q', t'\rangle = i\langle q'', t''|p(t'')|q', t'\rangle \tag{28a}$$

$$\frac{\partial}{\partial t''}\langle q'', t''|q', t'\rangle = -i\langle q'', t''|H(t'')|q', t'\rangle \tag{28b}$$

$$\frac{\partial}{\partial q'}\langle q'', t''|q', t'\rangle = -i\langle q'', t''|p(t')|q', t'\rangle \tag{28c}$$

$$\frac{\partial}{\partial t'}\langle q'', t''|q', t'\rangle = i\langle q'', t''|H(t')|q', t'\rangle \tag{28d}$$

In order to be able to integrate the system (28) we have to make sure that the six integrability conditions which correspond to all possible crossed second-order partial derivatives are satisfied. This is indeed the case as can be verified using

$$\frac{\partial}{\partial q}\langle q, t| = i\langle q, t|p(t) \tag{29a}$$

$$\frac{\partial}{\partial t}\langle q, t| = -i\langle q, t|H(t) \tag{29b}$$

which are just consequences of (28), together with the equations of motion

(15) for the operators. We present here the verificiation of $\partial^2 K/\partial q''\,\partial t'' = \partial^2 K/\partial t''\,\partial q''$ as an example of these calculations. From (28a) we obtain

$$\frac{\partial^2 K}{\partial t''\,\partial q''} = i\left(\frac{\partial}{\partial t''}\langle q'', t''|\right)p(t'')|q', t'\rangle + i\langle q'', t''|\dot{p}(t'')|q', t'\rangle$$

$$= \langle q'', t''|p(t'')H(t'')|q', t'\rangle \tag{30}$$

after using (29b) and the equation of motion for the operator p at $t = t''$. Similarly, from (28b) we can write

$$\frac{\partial^2 K}{\partial q''\,\partial t''} = -i\frac{\partial}{\partial q''}\langle q'', t''|H(t'')|q', t'\rangle$$

$$= \langle q'', t''|p(t'')H(t'')|q', t'\rangle \tag{31}$$

where we have used (29a) for $q = q''$. The remaining integrability conditions can be verified in an analogous way and they are automatically satisfied by virtue of the equations of motion and the properties (29).

Before closing this section we are going to show that the basic general relations between integrals of the motion and the Green's function found by Dodonov et al. (1975) are also a direct consequence of equation (6) and the explicit form of the generators (12). The integrals of the motion that Dodonov et al. (1975) consider are the initial position and momentum operators: $q(t')$ and $p(t')$ in our notation. From equation (27) we already derived the expression (28c) which expresses the partial derivative of the propagator with respect to the initial coordinate q' as a matrix element of the integral of motion $p(t')$. Now, this integral of motion can be written in principle as a function of the dynamical variables at any later time, in particular as a function of $q(t'')$ and $p(t'')$. Writing $p(t') = \hat{P}(q(t''), p(t''))$ and recalling (29a) we obtain the relation

$$\hat{P}\left(q'', \frac{1}{i}\frac{\partial}{\partial q''}\right)\langle q'', t''|q', t'\rangle = i\frac{\partial}{\partial q'}\langle q'', t''|q', t'\rangle \tag{32}$$

which is equation (28b) of the work of Dodonov et al. (1975). The action of the other constant of motion $q(t') = \hat{Q}(q(t''), p(t''))$ on the propagator is obtained after Fourier transforming an expression analogous to equation (28c) for the Green's function $\langle q'', t''|p', t'\rangle$ expressed in a mixed basis. In fact, the calculation of the kinematical variations of $\langle q'', t''|p', t'\rangle$ is given by

$$\delta\langle q'',t''|p',t'\rangle = i\langle q'',t''|p(t'')\,\delta q'' - H(t'')\,\delta t'' + q(t')\,\delta p' + H(t')\,\delta t'|p',t'\rangle \tag{33}$$

which incorporates the statement that $-q(t')\,\delta p'$ is the generator of variations in the momentum. From equation (33) we readily obtain

$$\frac{\partial}{\partial p'}\langle q'',t''|p',t'\rangle = i\langle q'',t''|q(t')|p',t'\rangle \tag{34}$$

which is the analogous of expression (28c). Inserting a complete set of states $|q',t'\rangle$ and recalling that $\langle q',t'|p',t'\rangle = (2\pi)^{1/2}\exp(ip'q')$ we can rewrite equation (34) as

$$\int_{-\infty}^{+\infty} dq'\,e^{ip'q'}\langle q'',t''|q',t'\rangle q' = \int_{-\infty}^{+\infty} dq'\,e^{ip'q'}\langle q'',t''|q(t')|q',t'\rangle \tag{35}$$

Taking the Fourier transform we obtain

$$\hat{Q}\left(q'',\frac{1}{i}\frac{\partial}{\partial q''}\right)\langle q'',t''|q',t'\rangle = q'\langle q'',t''|q',t'\rangle \tag{36}$$

which is equation (2.7a) of the work of Dodonov et al. (1975). As remarked by these authors, conditions (32) and (36) are not enough to completely determine the transformation function. In fact what is lacking is the information concerning the time evolution of the Green's function, which they incorporate through the Schrödinger equation corresponding to the final time parameter. Again we see that this information is already contained in (27) in the form of (28b)

$$\frac{\partial}{\partial t''}\langle q'',t''|q',t'\rangle = -i\langle q'',t''|H(t'')|q',t'\rangle = -iH\left(q'',\frac{1}{i}\frac{\partial}{\partial q''}\right)\langle q'',t''|q',t'\rangle \tag{37}$$

3. THE PROPAGATOR OF A DAMPED-FORCED HARMONIC OSCILLATOR WITH TIME-DEPENDENT PARAMETERS

3.1. Zero External Force. We take the system considered as the quantum analog of a classical damped harmonic oscillator to be described by the Hamiltonian

$$H = \tfrac{1}{2}\left[e^{-2\Gamma}p^2 + \omega^2 e^{2\Gamma}q^2\right] \tag{38}$$

where the frequency ω and the damping coefficient Γ are arbitrary functions of time.

The calculation of the propagator will proceed along the same lines described in the last section for a simple harmonic oscillator. In the first place we obtain the operator equations of motion

$$\begin{aligned} \dot{q} &= e^{-2\Gamma} p \\ \dot{p} &= -\omega^2 e^{2\Gamma} q \end{aligned} \tag{39}$$

which lead to

$$\ddot{q} + 2\dot{\Gamma}\dot{q} + \omega^2 q^2 = 0 \tag{40}$$

for the position operator. Again, the linearity of (40) allows us to write its general solution as

$$q(t) = \alpha(t) q'' + \beta(t) q' \tag{41}$$

where we are using the notation $q'' = q(t'')$ and $q' = q(t')$ for the operators. The auxiliary functions α and β are two independent solutions of the numerical equation

$$\ddot{z} + 2\dot{\Gamma}\dot{z} + \omega^2 z = 0 \tag{42}$$

which satisfy the following boundary conditions:

$$\begin{aligned} \alpha(t'') &= 1, \qquad \alpha(t') = 0 \\ \beta(t'') &= 0, \qquad \beta(t') = 1 \end{aligned} \tag{43}$$

dictated by our need of explicitly introducing the operators q' and q'' in the solution $q(t)$. From the first equation (39) we obtain

$$p(t) = e^{2\Gamma}(\dot{\alpha} q'' + \dot{\beta} q') \tag{44}$$

In the future we will need the commutator $[q'', q']$ which can be calculated in two alternative ways starting either from $[q', p'] = i$ or $[q'', p''] = i$ and using (44). The result is

$$[q'', q'] = i \frac{e^{-2\Gamma''}}{\dot{\beta}''} = -i \frac{e^{-2\Gamma'}}{\dot{\alpha}'} \tag{45}$$

Here the notation is $\Gamma'' = \Gamma(t'')$, $\dot{\beta}'' = d\beta/dt|_{t=t''}$ and similarly for Γ' and $\dot{\alpha}'$.

Of course the two equivalent expressions for the commutator (45) are just a consequence of the equations of motion as will be shown in the Appendix.

Now we are ready to calculate the change $\delta\langle q'', t''|q', t'\rangle$ according to (27). Using the Hamiltonian (38) together with the expressions (41), (44), and (45) we obtain

$$\begin{aligned}
&\delta \ln\langle q'', t''|q', t'\rangle \\
&\quad = i\Bigg[e^{2\Gamma''}\dot{\alpha}''\delta\left(\frac{q''^2}{2}\right) - e^{2\Gamma'}\dot{\beta}'\delta\left(\frac{q'^2}{2}\right) + e^{2\Gamma''}\dot{\beta}''\delta(q''q') \\
&\qquad - \frac{\delta t''}{2}\left(e^{2\Gamma''}\left(q''^2(\dot{\alpha}''^2 + \omega''^2) + q'^2\dot{\beta}''^2 + 2q'q''\dot{\alpha}''\dot{\beta}''\right) - i\dot{\alpha}''\right) \\
&\qquad + \frac{\delta t'}{2}\left(e^{2\Gamma'}\left(q''^2\dot{\alpha}''^2 + q'^2(\dot{\beta}'^2 + \omega'^2) + 2q'q''\dot{\alpha}'\dot{\beta}'\right) + i\dot{\beta}'\right)\Bigg]
\end{aligned} \tag{46}$$

where q' and q'' refer here to the eigenvalues of the respective operators.

The coordinate independent terms in (46) arise from the commutator (45) and provide the truly quantum contributions to the calculation.

The next step is to integrate expression (46). From the general discussion following (26) we know that the integrability conditions are automatically satisfied in virtue of the kinematics and the equations of motion. Taking advantage of this, we need to pay attention only to the first three terms together with the two coordinate independent terms of the right-hand side of (46) which can be readily integrated with respect to the coordinates and time, respectively. In the Appendix we verify that the appropriate time variations of the contributions coming from the coordinate integration do indeed correspond to the remaining terms in (46). In the Appendix we also show that

$$\begin{aligned}
\dot{\alpha}'' &= -\frac{\partial}{\partial t''}\ln\left(-\dot{\beta}''e^{2\Gamma''}\right) \\
\dot{\beta}' &= -\frac{\partial}{\partial t'}\ln\left(\dot{\alpha}'e^{2\Gamma'}\right) \\
\dot{\alpha}'e^{2\Gamma'} &= -\dot{\beta}''e^{2\Gamma''}
\end{aligned} \tag{47}$$

which permit us to integrate in general the coordinate independent terms of

(46). Our final result for the Green's function is then

$$\langle q'', t''|q', t'\rangle = \left(\frac{-\beta'' e^{2\Gamma''}}{2\pi i}\right)^{1/2} \times \exp\frac{i}{2}\left[\dot{\alpha}'' e^{2\Gamma''} q''^2 - \dot{\beta}' e^{2\Gamma'} q'^2 + 2e^{2\Gamma''}\dot{\beta}'' q'' q'\right] \quad (48)$$

where the numerical constant has been fixed by comparison with the simple harmonic oscillator case.

3.2. Addition of the External Force. In this section we complete the calculation of the forced, damped harmonic oscillator Green's function. The full Hamiltonian is now

$$H = \tfrac{1}{2}\left[e^{-2\Gamma}p^2 + \omega^2 e^{2\Gamma} q^2\right] - qe^{2\Gamma}F \quad (49)$$

where the parameters ω and Γ as well as the external force F are time dependent. The corresponding equations of motion are

$$\begin{aligned} \dot{q} &= e^{-2\Gamma}p \\ \dot{p} &= -\omega^2 e^{2\Gamma} q + e^{2\Gamma} F \end{aligned} \quad (50)$$

which yield

$$\ddot{q} + 2\dot{\Gamma}\dot{q} + \omega^2 q = F \quad (51)$$

for the position operator $q(t)$.

Let us remind ourselves that in Section 3.1 we have already found the propagator for the $F = 0$ case. The action principle allows us to take full advantage of this calculation by focusing our attention in the dynamical changes that arise in the transformation function when we vary the external force. In fact, we have

$$\begin{aligned} \delta_F\langle q'', t''|q', t'\rangle &= i\langle q'', t''|\delta_F \int_{t'}^{t''} L\,dt|q', t'\rangle \\ &= i\langle q'', t''|\int_{t'}^{t''} dt\, q(t) e^{2\Gamma(t)} \delta F(t)|q', t'\rangle \end{aligned} \quad (52)$$

where the change of the Lagrangian coming from the external force arises only from the last term in the Hamiltonian (49) and $q(t)$ satisfies equation

(51). The idea now is to reduce (52) to a numerical equation and to carry the integration with respect to $F(t)$ under the condition that $\langle q'', t''|q', t'\rangle_{F=0}$ is given by (48). Again the first step is to solve equation (51) introducing the operators q'' and q' as boundary conditions. This can be done recalling that the complete solution of (51) is the general solution of the related homogeneous equation, which we know from Section 3.1, plus any particular solution of (51). Thus we can write

$$q(t) = q''\alpha(t) + q'\beta(t) + \int_{t'}^{t''} G(t,\tau)F(\tau)\,d\tau \tag{53}$$

where α and β are given by (42) and (43).

The function $G(t,\tau)$ obeys

$$\left(\frac{d^2}{dt^2} + 2\dot{\Gamma}\frac{d}{dt} + \omega^2\right)G(t,\tau) = \delta(t-\tau) \tag{54}$$

and we must further require

$$G(t'',\tau) = G(t',\tau) = 0 \tag{55}$$

in order that (53) have the proper boundary conditions at t'' and t'. As it is well known, $G(t,\tau)$ can be constructed from two independent solutions of the corresponding homogeneous equation paying attention to the boundary conditions at $t=\tau$,

$$G(\tau-\varepsilon,\tau) = G(\tau+\varepsilon,\tau)$$

$$\left.\frac{dG}{dt}\right|_{t=\tau+\varepsilon} - \left.\frac{dG}{dt}\right|_{t=\tau-\varepsilon} = 1 \tag{56}$$

with $\varepsilon \to 0^+$. Condition (55) suggests the choice

$$G(t,\tau) = A\beta(t)\alpha(\tau)e^{2\Gamma(\tau)}, \qquad t>\tau \tag{57}$$

$$G(t,\tau) = A\alpha(t)\beta(\tau)e^{2\Gamma(\tau)}, \qquad t<\tau \tag{58}$$

The factor $e^{2\Gamma(\tau)}$ has been added to make sure that A is a numerical constant independent of τ. Continuity at $t=\tau$ is already built in because of the choices (57), (58). The discontinuity in the derivatives determine the constant A to be

$$A = \left[(\alpha(\tau)\dot{\beta}(\tau) - \beta(\tau)\dot{\alpha}(\tau))e^{2\Gamma(\tau)}\right]^{-1} \tag{59}$$

which is indeed independent of τ according to equation (A6) of the

Appendix. Now we can substitute (53) in (52) to obtain

$$\delta_F \ln\langle q'', t''|q', t'\rangle = i\left\{\delta_F \int_{t'}^{t''} dt\, e^{2\Gamma(t)} F(t)\left[q''\alpha(t) + q'\beta(t)\right]\right.$$
$$\left. + \int_{t'}^{t''} dt \int_{t'}^{t''} d\tau\, \delta F(t) e^{2\Gamma(t)} G(t,\tau) F(\tau)\right\} \quad (60)$$

The first term on the right-hand side of (60) can be readily integrated with respect to F while the second term needs still some transformations. Let us consider the combination $\mathscr{G} \equiv e^{2\Gamma(t)}G(t,\tau)$ which appears naturally in the double integral of (60)

$$\mathscr{G}(t,\tau) = A e^{2\Gamma(t)}\beta(t)\alpha(\tau)e^{2\Gamma(\tau)}, \qquad t > \tau$$
$$\mathscr{G}(t,\tau) = A e^{2\Gamma(t)}\alpha(t)\beta(\tau)e^{2\Gamma(\tau)}, \qquad t < \tau \quad (61)$$

From the above equations we see that $\mathscr{G}$ is symmetrical in its arguments: $\mathscr{G}(t,\tau) = \mathscr{G}(\tau,t)$. Because of this we can rewrite the double integral as

$$\int_{t'}^{t''} dt \int_{t'}^{t''} d\tau\, \delta F(t)\mathscr{G}(t,\tau)F(\tau) = \delta_F\left[\frac{1}{2}\int_{t'}^{t''} dt \int_{t'}^{t''} d\tau\, F(t)\mathscr{G}(t,\tau)F(\tau)\right] \quad (62)$$

which permits the integration with respect to F of this term. The final answer for the propagator is

$$\langle q'', t''|q', t'\rangle = \langle q'', t''|q', t'\rangle_{F=0}$$
$$\exp i\left[\int_{t'}^{t''} dt\, e^{2\Gamma(t)}F(t)\left(q''\alpha(t) + q'\beta(t)\right)\right.$$
$$\left. + \frac{1}{2}\int_{t'}^{t''} d\tau \int_{t'}^{t''} d\tau'\, F(\tau)\mathscr{G}(\tau,\tau')F(\tau')\right] \quad (63)$$

where $\langle q'', t''|q', t'\rangle_{F=0}$ is given by the expression (48) and $\mathscr{G}(\tau,\tau')$ by (61).

3.3. Comparison with Some Recent Calculations. In the first place we consider the work of Dodonov et al. (1979). Contact between their formulation and ours is established by means of the functions λ_1 and λ_3 which they define in equation (5) of their paper. As can be directly verified, such functions satisfy the differential equation (42) with boundary conditions

$$\lambda_1' = 1, \qquad \dot{\lambda}_1' = 0$$
$$\lambda_3' = 0, \qquad \dot{\lambda}_3' = -e^{2\Gamma'} \quad (64)$$

at the point $t' = 0$. Following the discussion in the Appendix we can express our functions α and β in terms of their functions λ_1 and λ_3 in the following way:

$$\alpha(t)=\frac{\lambda_3(t)}{\lambda_3''}$$

$$\beta(t)=\lambda_1(t)-\frac{\lambda_1''}{\lambda_3''}\lambda_3(t) \tag{65}$$

Using this realization we can calculate all the expressions required to compare the final results. We give the list of these equivalences for the terms appearing in our expression (48)

$$\dot{\beta}''e^{2\Gamma''}=\frac{1}{\lambda_3''}$$

$$\dot{\alpha}''e^{2\Gamma''}=\frac{\dot{\lambda}_3''e^{2\Gamma''}}{\lambda_3''} \tag{66}$$

$$\dot{\beta}'e^{2\Gamma'}=\frac{\lambda_1''}{\lambda_3''}$$

which exactly reproduce the corresponding terms in the result of Dodonov et al. (1979). Now we compare the $F(t)$ dependent terms given by our expression (63). The contributions which are linear in q'' and q' are directly reproduced after the substitution (67) is made. The coordinate independent term related to the double integral in (63) needs a bit more of algebraic manipulations which lead to

$$\frac{1}{2}\int_{t'}^{t''}d\tau\int_{t'}^{t''}d\tau' F(\tau)\mathscr{G}(\tau,\tau')F(\tau')$$
$$=-\frac{iA}{2\lambda_3''}\left[\frac{\lambda_1''}{\lambda_3''}\left(\int_{t_1}^{t''}d\tau e^{2\Gamma}\lambda_3 F\right)^2-2\int_{t'}^{t''}d\tau e^{2\Gamma}\lambda_1 F\int_{t'}^{\tau}d\tau' e^{2\Gamma}\lambda_3 F\right] \tag{67}$$

The constant A is calculated according to the definition (59) giving

$$A=\lambda_3'' \tag{68}$$

This completes the proof of the equivalence between our result (63) and that of Dodonov et al. (1979) for the transformation function of a damped, forced harmonic oscillator with time-dependent parameters.

Finally we comment on the result of Khandekar and Lawande (1979) and we briefly indicate how it is obtained starting from our expression (63). Their calculation is for the particular case $\Gamma(t)=(1/2)rt$ with r constant. They have chosen the auxiliary functions $\rho(t)$ and $\mu(t)$ to present their result. Such functions are the generalization of the amplitude and the phase, respectively, for a time-dependent oscillator and satisfy the following differential equations:

$$\ddot{\rho}+\left(\omega^2-\frac{r^2}{4}\right)\rho-\frac{1}{\rho^3}=0$$

$$\rho^2\dot{\mu}=1 \tag{69}$$

with arbitrary boundary conditions. Again, to establish contact between their expression for the propagation function and ours we only need to express our functions α and β in terms of their auxiliary functions ρ and μ. This is done by the relations

$$\alpha(t)=e^{(-r/2)(t-t'')}\frac{\rho(t)\sin[\mu(t)-\mu']}{\rho''\sin(\mu''-\mu')}$$

$$\beta(t)=e^{-(r/2)(t-t')}\frac{\rho(t)\sin[\mu''-\mu(t)]}{\rho'\sin(\mu''-\mu')} \tag{70}$$

It can be verified that the functions $e^{-rt/2}\rho(t)(\sin\mu(t),\cos\mu(t))$ satisfy the linear differential equation (42) (with $\dot{\Gamma}=r/2$) in virtue of the equations (69). Besides, the constants in (70) are so adjusted that the boundary conditions (43) are indeed satisfied. In this way we see that α and β defined in (70) are uniquely determined as solutions of a linear differential equation with given boundary conditions and consequently they cannot depend on the arbitrary boundary conditions associated to (69). It is precisely in equations (70) where we see that such boundary condition enter into the problem in a way that effectively cancels out. After some simple but tedious algebraic manipulations (Hernández, 1983) the substitution of expression (70) in our final result (63) for the propagator leads to the formula given by Khandekar and Lawande (1978) with the misprint already noticed by Dodonov et al. (1979) adequately corrected. We emphasize that our way of obtaining the result of Khandekar and Lawande clearly shows that this result is in fact independent of any boundary condition related to the functions ρ and μ, which is physically very satisfactory.

4. SUMMARY AND DISCUSSION

In this work we show how to calculate the propagator for the quantum mechanical analog of a damped, forced harmonic oscillator with arbitrary time-dependent parameters using a direct application of Schwinger action principle. This particular way of exploiting the action principle is successful because the operator part of the equations of motion for the coordinate and momentum operators can be made explicit in terms of the boundary conditions, for the system discussed here. This fact is a general feature of systems described by quadratical Hamiltonians and in particular the same method can be applied to work out the Green's function of the general damped, forced oscillator in the so-called coherent states representation.

Our final result for the propagator $K(q'', t''; q', t') = \langle q'', t''|q', t'\rangle$, given in (63) and (48), has an explicit dependence on the initial (q') and final (q'') coordinates. The time dependence is implicitly determined by the auxiliary functions $\alpha(t)$ and $\beta(t)$ which are independent solutions of the linear second-order differential equation (42) with boundary conditions given by (43). The explicit form of these uniquely defined auxiliary functions can be found in principle once the frequency $\omega(t)$ and the damping coefficient $\Gamma(t)$ are given.

We also make contact with the approach of Dodonov et al. (1975) for calculating quantum mechanical propagators which is based on the action of some integrals of the motion, taken to be the initial coordinate $q(t')$ and momentum $p(t')$ operators in this case, on the Green's function. We show how these general relations are obtained from the action principle and in particular we recover the result of Dodonov et al. (1979) for the damped, forced harmonic oscillator propagator using a particular representation of the functions α and β. This calculation constitutes and independent verification of the result of Dodonov et al. (1979). In their language we would say that our calculation considers the initial ($q(t')$) and final ($q(t'')$) position operator as integrals of the motion. This choice is as good as any other and is dictated here by the end points characterization of the propagator.

Finally, we briefly mention how the realization of α and β given in (70) allows us to recover the expression for the propagator derived by Khandekar and Lawande, for the particular case $\Gamma = (1/2)rt$, using a completely different approach. We also clarify a point concerning the apparent dependence of this expression on some unphysical boundary conditions.

APPENDIX

Here we discuss some points raised in Section 3.1 where we have introduced the auxiliary functions $\alpha(t; t', t'')$ and $\beta(t; t', t'')$ which obey the linear differential equation

$$\ddot{z} + 2\dot{\Gamma}(t)\dot{z} + \omega^2(t)z = 0 \tag{A1}$$

with the following boundary conditions:

$$\begin{aligned} \alpha|_{t=t''} &= 1, \qquad \alpha|_{t=t'} = 0 \\ \beta|_{t=t''} &= 0, \qquad \beta|_{t=t'} = 1 \end{aligned} \tag{A2}$$

We have written α and β as functions of t'' and t' to emphasize the fact that the boundary conditions are taken at such points.

In order to verify most of the statements made in section 3.1 we will need to calculate expressions like

$$\frac{\partial}{\partial t'}\left(\frac{d\alpha}{dt}\right)_{t=t''}, \qquad \frac{\partial}{\partial t''}\left(\frac{d\alpha}{dt}\right)_{t=t''}$$

and similarly for β. We see that the dependence on the argument with respect to which the derivative is taken comes from two places: (i) the original dependence upon the boundary conditions and (ii) the specific evaluation of the variable t at either t'' or t'. In order to be able to take derivatives it is very convenient to make such dependences more explicit. To this end we introduce two new independent auxiliary functions σ_i ($i = 1, 2$). We require them to satisfy equation (A1) with arbitrary boundary conditions chosen at any points different from t'' and t'. Such boundary conditions need not to be given in any detail and it is only enough that they properly define the functions σ_i. Then we can write our functions α and β as

$$\alpha(t) = \frac{1}{\Lambda}\left(\sigma_1'\sigma_2(t) - \sigma_2'\sigma_1(t)\right) \tag{A3a}$$

$$\beta(t) = \frac{1}{\Lambda}\left(\sigma_2''\sigma_1(t) - \sigma_1''\sigma_2(t)\right) \tag{A3b}$$

with

$$\Lambda = \sigma_2''\sigma_1' - \sigma_2'\sigma_1'' \tag{A4}$$

The notation is $\sigma_i' = \sigma_i(t')$, $\sigma_i'' = \sigma_i(t'')$ and the dependence of α and β on t'' and t' is now explicit.

Clearly, α and β defined in (A3) satisfy the differential equation (A1) because they are linear combinations of other solutions of this equation. Moreover, the appropriate boundary conditions (A2) are automatically incorporated by the definitions (A3) irrespectively of the boundary conditions used to specify the functions σ_i. More importantly, α and β given in (A3) must be independent of the specific boundary conditions which determine σ_i. This is because a solution of a linear differential equation like (A1) is uniquely determined by the boundary conditions which in this case correspond by construction to equations (A2). The functions σ_i have the property

$$\dot{\sigma}_i'' = \left.\frac{d\sigma_i(t)}{dt}\right|_{t=t''} = \frac{d}{dt''}\sigma_i(t'') \tag{A5}$$

and similarly for $t = t'$ because there is no implicit dependence on t'' or t'.

To begin with, we prove the equivalence of the two alternative ways of calculating $[q'', q']$ in (45). This is a direct consequence of the property

$$(\sigma_1(t)\dot{\sigma}_2(t) - \sigma_2(t)\dot{\sigma}_1(t))e^{2\Gamma(t)} = C \tag{A6}$$

where C is a numerical constant. Equation (A6) can be proved by taking the time derivative of the left-hand side and by using the equations of motion (A1) for σ_i. Using (A3a) we calculate

$$\dot{\alpha}' = \frac{1}{\Lambda}\left(\sigma_1'\dot{\sigma}_2' - \sigma_2'\dot{\sigma}_1'\right) = \frac{C}{\Lambda}e^{-2\Gamma'} \tag{A7}$$

where we have used (A6) in the last step. Analogously, from (A3b) we obtain

$$\dot{\beta}'' = \frac{1}{\Lambda}\left(\sigma_2''\dot{\sigma}_1'' - \sigma_1''\dot{\sigma}_2''\right) = -\frac{C}{\Lambda}e^{-2\Gamma''} \tag{A8}$$

The comparison between (A7) and (A8) yields the desired relation

$$\dot{\alpha}'e^{2\Gamma'} = -\dot{\beta}''e^{2\Gamma''} \tag{A9}$$

Now we obtain the relations (47) which allow us to perform the coordinate independent integrations in (46). Starting from (A7) we can write

$$\frac{\partial}{\partial t'}\ln(\dot{\alpha}'e^{2\Gamma'}) = -\frac{1}{\Lambda}\frac{\partial\Lambda}{\partial t'} = -\frac{1}{\Lambda}\left(\sigma_2''\frac{d\sigma_1'}{dt'} - \sigma_1''\frac{d\sigma_2'}{dt'}\right) \tag{A10}$$

where we have taken advantage of the explicit dependence of Λ on t' given by (A4). Using the property (A5) and recalling that

$$\dot{\beta}' = \frac{1}{\Lambda}\left(\sigma_2''\dot{\sigma}_1' - \sigma_1''\dot{\sigma}_2'\right) \tag{A11}$$

from (A3b), we obtain

$$\frac{\partial}{\partial t'}\ln\left(\dot{\alpha}' e^{2\Gamma'}\right) = -\dot{\beta}' \tag{A12}$$

Starting from (A8) we can show that

$$\frac{\partial}{\partial t''}\ln\left(-\dot{\beta}'' e^{2\Gamma''}\right) = -\dot{\alpha}'' \tag{A13}$$

in a completely analogous way. The point here is that the quantity being differentiated in (A12) and (A13) is the same, by virtue of (A9).

Finally we list all the partial derivatives which are needed to show that the variations with respect to t'' and t' of the coordinate dependent terms in the propagator (48) correspond to those exhibited in (46):

$$\frac{\partial}{\partial t'}\left(\dot{\alpha}'' e^{2\Gamma''}\right) = -\dot{\alpha}'\dot{\beta}'' e^{2\Gamma''} = \dot{\alpha}'^2 e^{2\Gamma'} \tag{A14}$$

$$\frac{\partial}{\partial t''}\left(\dot{\alpha}'' e^{2\Gamma''}\right) = -e^{2\Gamma''}\left(\dot{\alpha}''^2 + \omega''^2\right) \tag{A15}$$

$$\frac{\partial}{\partial t'}\left(\dot{\beta}' e^{2\Gamma'}\right) = -e^{2\Gamma'}\left(\dot{\beta}'^2 + \omega'^2\right) \tag{A16}$$

$$\frac{\partial}{\partial t''}\left(\dot{\beta}' e^{2\Gamma'}\right) = -\dot{\alpha}'\dot{\beta}'' e^{2\Gamma'} = e^{2\Gamma''}\dot{\beta}''^2 \tag{A17}$$

$$\frac{\partial}{\partial t'}\left(\dot{\beta}'' e^{2\Gamma''}\right) = \dot{\alpha}'\dot{\beta}' e^{2\Gamma'} \tag{A18}$$

$$\frac{\partial}{\partial t''}\left(\dot{\beta}'' e^{2\Gamma''}\right) = -\dot{\alpha}''\dot{\beta}'' e^{2\Gamma''} \tag{A19}$$

All calculations are straightforward applications of the representation (A3) for α and β together with the use of the equations of motion (A1) and the relation (A9). In particular (A18) and (A19) an just another way of writing (A12) and (A13), respectively.

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